International Journal of Theoretical Physics, Vol. 24, No. 10, 1985

Saalschutzians and Racah Coefficients

K. Srinivasa Rao[1] and V. Rajeswari[1]

Received July 27, 1984

The set I and set II of three and four ${}_4F_3(1)s$ for the Racah coefficient are shown to be related through the property of reversal of series of the generalized Saalschutzian hypergeometric function.

Recently there has been considerable interest in the connection between angular momentum coupling coefficients (Raynal, 1978; Srinivasa Rao, 1978)/recoupling coefficients (Srinivasa Rao, Santhanam and Venkatesh, 1975; Srinivasa Rao and Venkatesh, 1975: Raynal, 1979; Wilson, 1980; Biedenharn and Louck, 1981a) and generalized hypergeometric functions of unit argument. In this note, we show that the set I of three ${}_4F_3(1)s$ for the $6-j$ coefficient (Srinivasa Rao, Santhanam, and Venkatesh, 1975) are related to the set II of four ${}_4F_3(1)s$ for the $6-j$ coefficient (Srinivasa Rao and Venkatesh, 1977) through the property of "reversal of series" of the generalized Saalschutzian ${}_4F_3(1)s$. The conventional expansion for the Racah coefficient (see, for instance, Biedenharn and Louck, 1981) is given by

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{a+b+c+d} W(abcd; ef)$$

$$= N \sum_p (-1)^P (P+1)! \left[\prod_{i=1}^{4} (P-\alpha_i)! \prod_{j=1}^{3} (\beta_j - P)! \right]^{-1} \qquad (1)$$

where the range of p is

$$P_{\min} \leq P \leq P_{\max}$$

with $P_{\min} = \max(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $P_{\max} = \min(\beta_1, \beta_2, \beta_3)$,

$$\alpha_1 = a+b+e, \qquad \alpha_2 = c+d+e, \qquad \alpha_3 = a+c+f, \qquad \alpha_4 = b+d+f,$$

[1]MATSCIENCE, The Institute of Mathematical Sciences, MADRAS-600113, India.

$$N = \Delta(abe);\ \Delta(cde);\ \Delta(acf);\ \Delta(bdf)$$

$$\beta_1 = a+b+c+d, \qquad \beta_2 = a+d+e+f, \qquad \beta_3 = b+c+e+f$$

and

$$\Delta(XYZ) = [(X+Y-Z)!(X-Y+Z)!(-X+Y+Z)!/(X+Y+Z+1)!]^{1/2}$$

We have shown (Srinivasa Rao, Santhanam, and Venkatesh, 1975) that (1) can be rewritten in terms of a set of three Saalschutzian ${}_4F_3(1)s$ of unit argument, as

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{E+1} N\Gamma(1-E)[\Gamma(1-A, 1-B, 1-C, 1-D, F, G)]^{-1} \times {}_4F_3(ABCD;\ EFG;\ 1) \tag{2}$$

where

$$\Gamma(X, Y, Z, \ldots) = \Gamma(X)\cdot\Gamma(Y)\cdot\Gamma(Z)\ldots,$$

$$\begin{aligned} &A = e-a-b, \quad B = e-c-d, \quad C = f-c-a, \quad D = f-d-b \\ &E = -a-b-c-d-1, \quad F = e+f-a-d+1, \quad G = e+f-b-c+1 \end{aligned} \tag{3}$$

and the set of three ${}_4F_3(1)s$ is obtained by superposing the permutations of the columns of $\{\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\}$ on the given ${}_4F_3(1)$ in (3). Superposing the interchange of any two elements in a row of $\{\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\}$ with the corresponding elements in the other row only results in a permutation of the numerator and denominator parameters among themselves, in a given ${}_4F_3(1)$ belonging to this set.

By adopting a similar procedure we (Srinivasa Rao and Venkatesh, 1976) obtained an equivalent set II of four ${}_4F_3(1)s$ for (1) as

$$\begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = (-1)^{A'} N\Gamma(A')[\Gamma(1-B', 1-C', 1-D', E', F', G')]^{-1} \times {}_4F_3(A'B'C'D';\ E'F'G';\ 1) \tag{4}$$

where

$$\begin{aligned} &A' = a+c+f+2, \quad B' = c-d-e, \quad C' = a-b-e, \quad D' = f-b-d \\ &E' = a+c-b-d+1, \quad F' = a+f-d-e+1, \quad G' = c+f-b-e+1 \end{aligned} \tag{5}$$

and the set of four ${}_4F_3(1)s$ is spanned by superposing the interchange of any two elements in a row of $\{\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\}$ with the corresponding elements in the other row, on the given ${}_4F_3(1)$ in (5). Superposing the permutations of the columns of $\{\begin{smallmatrix} a & b & e \\ d & c & f \end{smallmatrix}\}$ on a given ${}_4F_3(1)$ belonging to this set results only in a permutation of the numerator and denominator parameters among themselves.

Since more than one numerator parameter in the set I and set II of ${}_4F_3(1)s$ is a negative integer, we can generalize the property of "reversal of the series" given by Bailey (1935) for the case of a ${}_3F_2(1)$, to the case of a Saalschutzian-${}_4F_3(1)$ to obtain

$$\begin{aligned} {}_4F_3(ABCD;\, EFG;\, 1) = {} & (-1)^D\Gamma(1-A, 1-B, 1-C, F, G, D-E+1) \\ & \times[\Gamma(D-A+1, D-B+1, D-C+1, F-D, \\ & \quad G-D, 1-E)]^{-1} \\ & \times{}_4F_3(D-E+1, D-F+1, D-G+1, D; \\ & \quad D-A+1, D-B+1, D-C+1; 1) \end{aligned} \tag{6}$$

where D is the negative parameter which determines the number of terms in the series.

Substituting the right-hand side of (6) for the ${}_4F_3(1)$ which occurs in equation (3), after some simple algebraic manipulations we get

$$\begin{aligned} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix} = {} & (-1)^{A'}N\Gamma(A')[\Gamma(F', G', E', 1-C', 1-B', 1-D')]^{-1} \\ & \times{}_4F_3(A'C'B'D';\, F'G'E';\, 1) \end{aligned} \tag{5$'$}$$

which is nothing but the ${}_4F_3(1)$ belonging to set II given in (5) except for a trivial permutation among the numerator and denominator parameters.

Thus, we find that the set I and set II of ${}_4F_3(1)s$ for the Racah coefficient are not independent but are related to one another by the property of reversal of series. It is intriguing to note that owing to this reversal of series property the set I of *three* ${}_4F_3(1)s$ yields the set II of *four* ${}_4F_3(1)s$ and vice-versa.

ACKNOWLEDGMENTS

The authors thank Professor R. Vasudevan, Dr. R. Sridhar, and Dr. Geetha S. Rao for interesting discussions. One of us (V. Rajeswari) thanks the Institute of Mathematical Sciences for the award of "Matscience-DAE" Junior Research Fellowship.

REFERENCES

Bailey, W. N. (1935). *Generalised Hypergeometric Series*, Cambridge University Press, Cambridge.

Biedenharn, L. C., and Louck, J. D. (1981a), *The Racah-Wigner Algebra in Quantum Theory, Encyclopedia of Mathematics and Its Applications*, Vol. 9. Addison-Wesley, London.

Biedenharn, L. C., and Louck, J. D. (1981). *Angular Momentum in Quantum Physics, Encyclopedia of Mathematics and Its Applications*, Vol. 8. Addison-Wesley, London.
Raynal, J. (1978). *Journal of Mathematical Physics*, **19**, 467.
Raynal, J. (1979). *Journal of Mathematical Physics*, **20**, 2398.
Srinivasa Rao, K. (1978). *Journal of Physics A: Mathematical and General*, **11**, L69.
Srinivasa Rao, K., Santhanam, T. S., and Venkatesh, K. (1975). *Journal of Mathematical Physics*, **16**, 1528.
Srinivasa Rao, K., and Venkatesh, K. (1976). *Proceedings of the V International Colloquium on Group Theoretical Methods in Physics, New York*. Academic Press, London.
Wilson, A. (1980). *Siam Journal of Mathematical Analysis*, **11**, 690.